

$gl_q(n)$ -Covariant Multimode Oscillators and q-Symmetric States

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Abstract

In this paper the coherent states and q-symmetric states for $gl_q(n)$ -covariant multimode oscillator system are investigated.

1 Introduction

Quantum groups or q-deformed Lie algebra implies some specific deformations of classical Lie algebras.

From a mathematical point of view, it is a non-commutative associative Hopf algebra. The structure and representation theory of quantum groups have been developed extensively by Jimbo [1] and Drinfeld [2].

The q-deformation of Heisenberg algebra was made by Arik and Coon [3], Macfarlane [4] and Biedenharn [5]. Recently there has been some interest in more general deformations involving an arbitrary real functions of weight generators and including q-deformed algebras as a special case [6-10].

Recently Greenberg [11] has studied the following q-deformation of multimode boson algebra:

$$a_i a_j^\dagger - q a_j^\dagger a_i = \delta_{ij},$$

where the deformation parameter q has to be real. The main problem of Greenberg's approach is that we can not derive the relation among a_i 's operators at all. Moreover the above algebra is not covariant under $gl_q(n)$ algebra. In order to solve this problem we should find the q -deformed multimode oscillator algebra which is covariant under $gl_q(n)$ algebra. Recently the Fock space representation of $gl_q(n)$ -covariant multimode oscillator system was known by some authors [12]. In this paper we construct the correct form of coherent states for the above mentioned oscillator system and obtain the q -symmetric states generalizing the bosonic states.

2 Coherent states of $gl_q(n)$ -covariant oscillator algebra

$gl_q(n)$ -covariant oscillator algebra is defined as [12]

$$\begin{aligned}
a_i^\dagger a_j^\dagger &= q a_j^\dagger a_i^\dagger, \quad (i < j) \\
a_i a_j &= \frac{1}{q} a_j a_i, \quad (i < j) \\
a_i a_j^\dagger &= q a_j^\dagger a_i, \quad (i \neq j) \\
a_i a_i^\dagger &= 1 + q^2 a_i^\dagger a_i + (q^2 - 1) \sum_{k=i+1}^n a_k^\dagger a_k, \quad (i = 1, 2, \dots, n-1) \\
a_n a_n^\dagger &= 1 + q^2 a_n^\dagger a_n, \\
[N_i, a_j] &= -\delta_{ij} a_j, \quad [N_i, a_j^\dagger] = \delta_{ij} a_j^\dagger, \quad (i, j = 1, 2, \dots, n)
\end{aligned} \tag{1}$$

where we restrict our concern to the case that q is real and $0 < q < 1$. Here N_i plays a role of number operator and $a_i(a_i^\dagger)$ plays a role of annihilation(creation) operator. From the above algebra one can obtain the relation

between the number operators and mode operators as follows

$$a_i^\dagger a_i = q^{2\sum_{k=i+1}^n N_k} [N_i], \quad (2)$$

where $[x]$ is called a q-number and is defined as

$$[x] = \frac{q^{2x} - 1}{q^2 - 1}.$$

Let us introduce the Fock space basis $|n_1, n_2, \dots, n_n\rangle$ for the number operators N_1, N_2, \dots, N_n satisfying

$$N_i |n_1, n_2, \dots, n_n\rangle = n_i |n_1, n_2, \dots, n_n\rangle, \quad (n_1, n_2, \dots, n_n = 0, 1, 2, \dots) \quad (3)$$

Then we have the following representation

$$\begin{aligned} a_i |n_1, n_2, \dots, n_n\rangle &= q^{\sum_{k=i+1}^n n_k} \sqrt{[n_i]} |n_1, \dots, n_i - 1, \dots, n_n\rangle \\ a_i^\dagger |n_1, n_2, \dots, n_n\rangle &= q^{\sum_{k=i+1}^n n_k} \sqrt{[n_i + 1]} |n_1, \dots, n_i + 1, \dots, n_n\rangle. \end{aligned} \quad (4)$$

From the above representation we know that there exists the ground state $|0, 0, \dots, 0\rangle$ satisfying $a_i |0, 0, \dots, 0\rangle = 0$ for all $i = 1, 2, \dots, n$. Thus the state $|n_1, n_2, \dots, n_n\rangle$ is obtained by applying the creation operators to the ground state $|0, 0, \dots, 0\rangle$

$$|n_1, n_2, \dots, n_n\rangle = \frac{(a_n^\dagger)^{n_n} \dots (a_1^\dagger)^{n_1}}{\sqrt{[n_1]! \dots [n_n]!}} |0, 0, \dots, 0\rangle. \quad (5)$$

If we introduce the scale operators as follows

$$Q_i = q^{2N_i}, \quad (i = 1, 2, \dots, n), \quad (6)$$

we have from the algebra (1)

$$[a_i, a_i^\dagger] = Q_i Q_{i+1} \dots Q_n. \quad (7)$$

Acting the operators Q_i 's on the basis $|n_1, n_2, \dots, n_n\rangle$ produces

$$Q_i |n_1, n_2, \dots, n_n\rangle = q^{2n_i} |n_1, n_2, \dots, n_n\rangle. \quad (8)$$

From the relation $a_i a_j = \frac{1}{q} a_j a_i$, ($i < j$), the coherent states for $gl_q(n)$ algebra is defined as

$$a_i |z_1, \dots, z_i, \dots, z_n\rangle = z_i |z_1, \dots, z_i, qz_{i+1}, \dots, qz_n\rangle. \quad (9)$$

Solving the eq.(9) we obtain

$$|z_1, z_2, \dots, z_n\rangle = c(z_1, \dots, z_n) \sum_{n_1, n_2, \dots, n_n=0}^{\infty} \frac{z_1^{n_1} z_2^{n_2} \dots z_n^{n_n}}{\sqrt{[n_1]![n_2]!\dots[n_n]!}} |n_1, n_2, \dots, n_n\rangle. \quad (10)$$

Using eq.(5) we can rewrite eq.(10) as

$$|z_1, z_2, \dots, z_n\rangle = c(z_1, \dots, z_n) \exp_q(z_n a_n^\dagger) \dots \exp_q(z_2 a_2^\dagger) \exp_q(z_1 a_1^\dagger) |0, 0, \dots, 0\rangle. \quad (11)$$

where q-exponential function is defined as

$$\exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!}.$$

The q-exponential function satisfies the following recurrence relation

$$\exp_q(q^2 x) = [1 - (1 - q^2)x] \exp_q(x) \quad (12)$$

Using the above relation and the fact that $0 < q < 1$, we obtain the formula

$$\exp_q(x) = \prod_{n=0}^{\infty} \frac{1}{1 - (1 - q^2)q^{2n}x} \quad (13)$$

Using the normalization of the coherent state, we have

$$c(z_1, z_2, \dots, z_n) = \exp_q(|z_1|^2) \exp_q(|z_2|^2) \dots \exp_q(|z_n|^2). \quad (14)$$

The coherent state satisfies the completeness relation

$$\int \cdots \int |z_1, z_2, \cdots, z_n\rangle \langle z_1, z_2, \cdots, z_n| \mu(z_1, z_2, \cdots, z_n) d^2 z_1 d^2 z_2 \cdots d^2 z_n = I, \quad (15)$$

where the weighting function $\mu(z_1, z_2, \cdots, z_n)$ is defined as

$$\mu(z_1, z_2, \cdots, z_n) = \frac{1}{\pi^2} \prod_{i=1}^n \frac{\exp_q(|z_i|^2)}{\exp_q(q|z_i|^2)}. \quad (16)$$

In deriving eq.(15) we used the formula

$$\int_0^{1/(1-q^2)} x^n \exp_q(q^2 x)^{-1} d_{q^2} x = [n]! \quad (17)$$

3 q-symmetric states

In this section we study the statistics of many particle state. Let N be the number of particles. Then the N -particle state can be obtained from the tensor product of single particle state:

$$|i_1, \cdots, i_N\rangle = |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_N\rangle, \quad (18)$$

where i_1, \cdots, i_N take one value among $\{1, 2, \cdots, n\}$ and the single particle state is defined by $|i_k\rangle = a_{i_k}^\dagger |0\rangle$.

Consider the case that k appears n_k times in the set $\{i_1, \cdots, i_N\}$. Then we have

$$n_1 + n_2 + \cdots + n_n = \sum_{k=1}^n n_k = N. \quad (19)$$

Using these facts we can define the q -symmetric states as follows:

$$|i_1, \cdots, i_N\rangle_q = \sqrt{\frac{[n_1]! \cdots [n_n]!}{[N]!}} \sum_{\sigma \in Perm} \text{sgn}_q(\sigma) |i_{\sigma(1)} \cdots i_{\sigma(N)}\rangle, \quad (20)$$

where

$$\text{sgn}_q(\sigma) = q^{R(i_1 \cdots i_N)} q^{R(\sigma(1) \cdots \sigma(N))},$$

$$R(i_1, \cdots, i_N) = \sum_{k=1}^N \sum_{l=k+1}^N R(i_k, i_l)$$

and

$$R(i, j) = \begin{cases} 1 & \text{if } i > j \\ 0 & \text{if } i \leq j \end{cases}$$

Then the q-symmetric states obeys

$$|\cdots, i_k, i_{k+1}, \cdots \rangle_q = \begin{cases} q^{-1} |\cdots, i_{k+1}, i_k, \cdots \rangle_q & \text{if } i_k < i_{k+1} \\ |\cdots, i_{k+1}, i_k, \cdots \rangle_q & \text{if } i_k = i_{k+1} \\ q |\cdots, i_{k+1}, i_k, \cdots \rangle_q & \text{if } i_k > i_{k+1} \end{cases} \quad (21)$$

The above property can be rewritten by introducing the deformed transition operator $P_{k,k+1}$ obeying

$$P_{k,k+1} |\cdots, i_k, i_{k+1}, \cdots \rangle_q = |\cdots, i_{k+1}, i_k, \cdots \rangle_q \quad (22)$$

This operator satisfies

$$P_{k+1,k} P_{k,k+1} = Id, \quad \text{so } P_{k+1,k} = P_{k,k+1}^{-1} \quad (23)$$

Then the equation (21) can be written as

$$P_{k,k+1} |\cdots, i_k, i_{k+1}, \cdots \rangle_q = q^{-\epsilon(i_k, i_{k+1})} |\cdots, i_{k+1}, i_k, \cdots \rangle_q \quad (24)$$

where $\epsilon(i, j)$ is defined as

$$\epsilon(i, j) = \begin{cases} 1 & \text{if } i > j \\ 0 & \text{if } i = j \\ -1 & \text{if } i < j \end{cases}$$

The relation (24) goes to the symmetric relation for the ordinary bosons when the deformation parameter q goes to 1. If we define the fundamental q -symmetric state $|q\rangle$ as

$$|q\rangle = |i_1, i_2, \dots, i_N\rangle_q$$

with $i_1 \leq i_2 \leq \dots \leq i_N$, we have

$$||q\rangle|^2 = 1.$$

In deriving the above relation we used following identity

$$\sum_{\sigma \in Perm} q^{2R(\sigma(1), \dots, \sigma(N))} = \frac{[N]!}{[n_1]! \dots [n_n]!} \quad (25)$$

The derivation of above formula will be given in Appendix.

4 Concluding Remark

In this paper the $gl_q(n)$ -covariant oscillator algebra and its coherent states are discussed. The q -symmetric states generalizing the symmetric (bosonic) states are obtained by using the $gl_q(n)$ -covariant oscillators and are shown to be orthonormal. I think that the q -symmetric states will be important when we consider the new statistical field theory generalizing the ordinary one.

Appendix

In this appendix we prove the relation(25) by using the mathematical induction. Let us assume that the relation (25) holds for N . Now we should prove

that eq.(25) still hold for $N+1$. Let us consider the case that i appears n_i+1 times. Then we should show

$$\sum_{\sigma \in Perm} q^{2R(\sigma(1), \dots, \sigma(N+1))} = \frac{[N+1]!}{[n_1]! \cdots [n_{i-1}]! [n_i+1]! [n_{i+1}]! \cdots [n_n]!} \quad (26)$$

In this case the above sum can be written by three pieces:

$$\sum_{j=1}^{i-1} \sum_{\sigma(1)=j} + \sum_{\sigma(1)=i} + \sum_{j=i+1}^n \sum_{\sigma(1)=j} \quad (27)$$

Thus the left hand side of eq.(26) is given by

$$\begin{aligned} LHS &= \sum_{j=1}^{i-1} \sum_{\sigma(1)=j} q^{2R(j, \sigma(2), \dots, \sigma(n+1))} \\ &+ \sum_{\sigma(1)=i} q^{2R(i, \sigma(2), \dots, \sigma(n+1))} \\ &+ \sum_{j=i+1}^n \sum_{\sigma(1)=j} q^{2R(j, \sigma(2), \dots, \sigma(n+1))} \end{aligned} \quad (28)$$

Then we have

$$R(j, \sigma(2), \dots, \sigma(n+1)) = \sum_{k=2}^{N+1} R(j, \sigma(k)) + R(\sigma(2), \dots, \sigma(N+1))$$

where

$$\sum_{k=2}^{N+1} R(j, \sigma(k)) = \begin{cases} n_1 + \cdots + n_{j-1} & \text{if } j \leq i \\ n_1 + \cdots + n_{j-1} + 1 & \text{if } j > i \end{cases}$$

Using the above relations the LHS of eq.(26) can be written as

$$\begin{aligned} LHS &= \sum_{j=1}^{i-1} q^{2(n_1 + \cdots + n_{j-1})} \frac{[N]!}{[n_i+1]! [n_j-1]! \Pi_{k \neq i, j} [n_k]!} \\ &+ q^{2(n_1 + \cdots + n_{i-1})} \frac{[N]!}{\Pi_k [n_k]!} \end{aligned}$$

$$(29) \quad + \sum_{j=i+1}^{N+1} q^{2(n_1+\dots+n_{j-1}+1)} \frac{[N]!}{[n_i+1]![n_j-1]!\prod_{k \neq i,j} [n_k]!}$$

If we pick up the common factor of three terms of eq.(29), we have

$$I = J \frac{[N]!}{[n_i+1]!\prod_{k \neq i} [n_k]!}$$

where

$$\begin{aligned} J &= \frac{1}{q^2-1} \left[\sum_{j=1}^{i-1} q^{2(n_1+\dots+n_{j-1})} [n_j] + q^{2(n_1+\dots+n_{i-1})} [n_i+1] + \sum_{j=i+1}^{N+1} q^{2(n_1+\dots+n_{j-1}+1)} [n_j] \right] \\ &= [N+1] \end{aligned} \quad (30)$$

Thus we proved the relation (25).

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References

- [1] Jimbo, Lett.Math.Phys.10 (1985) 63;11(1986)247.
- [2] V.Drinfeld, Proc.Intern.Congress of Mathematicians (Berkeley, 1986) 78.
- [3] M.Arik and D.Coon, J.Math.Phys.17 (1976) 524.
- [4] A.Macfarlane, J.Phys.A22(1989) 4581.
- [5] L.Biedenharn, J.Phys.A22(1989)L873.
- [6] A.Polychronakos, Mod.Phys.Lett.A5 (1990) 2325.
- [7] M.Rocek, Phys.Lett.B225 (1991) 554.
- [8] C.Daskaloyannis, J.Phys.A24 (1991) L789.
- [9] W.S.Chung, K.S.Chung, S.T.Nam and C.I.Um, Phys.Lett.A183 (1993) 363.
- [10] W.S.Chung, J.Math.Phys.35 (1994) 3631.
- [11] O.Greenberg, Phys.Rev.D43(1991)4111
- [12] R.Jagannathan, et.al., J.Phys.A25(1992) 6429.